FUNDAMENTAL SOLUTION OF A CAUCHY–POISSON PROBLEM FOR A BASIN WITH AN UNEVEN BOTTOM

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1. Consideration is given to a plane linear initial boundary problem

$$\Delta U = 0 \ (H(x) < y < 0), \ U_{tt} + U_y = 0 \ (y = 0),$$
$$U_y = H_x U_x \ (y = H(x)), \ U = 0, \ U_t = -\delta(x - x_0) \ (t = 0, \ y = 0), \tag{1.1}$$

which describes movement of a liquid caused by an initial disturbance of the free boundary. At instant of time t = 0 the liquid surface has a concentrated elevated area equal to one in the vicinity of point x_0, and with t > 0 this elevation breaks down under the force of gravity. Relationships (1.1) are written in dimensionless variables, and scales for length and velocity are selected such that the Froude number for the problem and depth of the liquid with $|x| \to \infty$ equal one. A right Cartesian coordinate system is orientated so that axis y is in a direction opposite to that of free fall; function $U(x, y, t, x_0)$ is the potential of velocities depending on x_0 as for the parameter; function H(x) describes the relief of the bottom.

Function $U(x, y, t, x_0)$ will be called the fundamental solution of the Cauchy–Poisson problem since with use of it solution of the general problem [1]

$$\Delta\varphi = 0 \ (H(x) < y < 0, \ x \in R^1), \ \varphi_{tt} + \varphi_y + p_t(x, t) = 0 \ (y = 0),$$
$$\varphi_y = H_x\varphi_x \ (y = H(x)), \ \varphi = \varphi_0(x), \ \varphi_t = \varphi_1(x) \ (t = 0, \ y = 0)$$

is described in quadratures

$$\varphi(x, y, t) = \int_{-\infty}^{\infty}\int_0^t U_t(x, y, t - t_0, x_0)\, p(x_0, t_0)\, dt_0\, dx_0 - \int_{-\infty}^{\infty} U_t(x, y, t, x_0)\,\varphi_0(x_0)\, dx_0 - \int_{-\infty}^{\infty} U(x, y, t, x_0)\,\varphi_1(x_0)\, dx_0.$$

In the future the case is considered when the depth of the basin changes slowly, i.e., $H(x) = -h(\varepsilon x)$ $\left(\varepsilon \ll 1, \ \max_{\xi \in R^1} |h_\xi| = 1\right)$.

The problem formulated belongs to a broad range of problems about propagation of a signal in an inhomogeneous medium with slowly changing properties. Currently two approaches are known for approximate solution of a problem of this sort: Keller [2], based on a notion of high-frequency asymptotics, and an approach recently intensively developed by Dobrokhotov and Zhevandrov [1, 3] based on Maslov methods. In [1] a solution of the formulated problem is built up in the three-dimensional case with accuracy prescribed beforehand with respect to parameter ε. The approximate solution is the sum of two terms, the first of which describes the long-wave component, and the second the short-wave component. The long-wave component is found from a recurrent sequence of problems for an inhomogeneous wave equation with a variable coefficient. The short-wave component of the solution is written out by means of quadratures. The equations obtained with this approach may be simplified by means of known methods of analyzing integrals depending on parameters and given in a form suitable for numerical calculations.

It is of interest to obtain these results by means of a special modification of the Keller method. In the present work a method is used of combined asymptotic expansions which make it possible to build up the main term of uniformly suitable asymptotics for deformation of the free boundary η(x, t) [the equation y = η(x, t) gives the shape of the free boundary

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at instant t, $\eta(x, t) = -U_t(x, 0, t, x_0)$] with $\varepsilon \to 0$, $t > 0$, $x \in R^1$. For the given problem these asymptotics may be written in explicit form, which makes it possible to carry out a detailed study.

Solution of problem (1.1) is found in the form of an asymptotic expansion for the power of small parameter ε

$$U(x, y, t, x_0) = \sum_{j=0}^{\infty} \varepsilon^j U^{(j)}(x_1, y, t, x_0)$$

($x_1 = x - x_0$). The principal term of this expansion describes solution of the Cauchy–Poisson problem for an even bottom $H(x) = -h_0 = -h(\varepsilon x_0)$. With $y = 0$ we have [4]

$$U^{(0)}(x_1, 0, t, x_0) = -\frac{1}{\pi}\int_0^{\infty} \Omega^{-1}(\nu) \sin \Omega(\nu) t \cos \nu x_1 \, d\nu \tag{1.2}$$

($\Omega^2(\nu) = \nu$ than νh_0). Asymptotics of function (1.2) with $x_1 = (\xi - \xi_0)/\varepsilon$, $t = \tau/\varepsilon$, $\varepsilon \to 0$, $\xi \sim 1$, $\xi_0 \sim 1$, $\tau \sim 1$ are written out by means of the stationary phase method [5]

$$U^{(0)} = \varepsilon^{1/2}\tau^{-1/2}(8\pi|\ddot{\Omega}(\alpha)|)^{-1/2}\Omega^{-1}(\alpha)\exp\left\{\frac{i}{\varepsilon}(\tau\Omega(\alpha) + \alpha(\xi - \xi_0)) + i\frac{\pi}{4}\right\} + O([\varepsilon/\tau]^{3/2}) + \text{c. c.} \tag{1.3}$$

($\alpha = \alpha((\xi - \xi_0)/\tau)$ is the solution of the equation $\dot{\Omega}(\alpha) = -(\xi - \xi_0)/\tau$). It is clear that the expansion selected is unsuitable with large values of x_1, t, since it does not make it possible to satisfy the nonflow condition at the bottom with the accuracy prescribed beforehand and uniformly with respect to x_1.

2. In order to refine the structure of the solution in region $|x_1| \gg 1$ new 'slow' variables $\xi = \varepsilon x$, $\tau = \varepsilon t$, $\xi_0 = \varepsilon x_0$ are introduced, and the asymptotic expansion of the solution is found in the form [2]

$$U(x, y, t, x_0) = \varepsilon^{1/2} e^{\frac{i}{\varepsilon}\theta(\xi,\tau,\xi_0)} \sum_{j=0}^{\infty} (i\varepsilon)^j A_j(\xi, y, \tau, \xi_0) + \text{c. c.} \tag{2.1}$$

This form of the expansion follows from limiting relationship (1.3) which should be continuous in accordance with the main term of expansion (2.1). Substitution of (2.1) in a Laplace equation and boundary conditions (1.1) in the normal way [2] leads to an equation for phase function θ and transfer equations for A_j.

In a zero approximation we have a spectral problem

$$\theta_\xi^2 A_0 + A_{0yy} = 0 \ (-h(\xi) < y < 0), \quad A_{0y} = 0 \ (y = -h(\xi)), \quad -\theta_\tau^2 A_0 + A_{0y} = 0 \ (y = 0),$$

whose neutral solution exists with the conditions

$$\theta_\tau^2 = \theta_\xi \ \text{th}\,[\theta_\xi h(\xi)] \tag{2.2}$$

and it is written as

$$A_0(\xi, y, \tau, \xi_0) = C_0(\xi, \tau, \xi_0) \,\text{ch}\, \theta_\xi(y + h(\xi)).$$

Equation (2.2) with the first order partial derivatives has a standard form $F(\xi, q, \omega) = 0$ ($q = \theta_\xi$, $\omega = \theta_\tau$, $F(\xi, q, \omega) = \omega^2 - q$ than $qh(\xi)$). It corresponds to a characteristic system [6]

$$\begin{gathered} d\omega/d\lambda = 0, \ d\tau/d\lambda = 2\omega, \ d\xi/d\lambda = -S(qh(\xi)), \ dq/d\lambda = q^2 h_\xi/(\text{ch}^2\, qh), \\ d\theta/d\lambda = 2\omega^2 - qS(qh(\xi)), \ S(x) = \text{th}\, x + x\, \text{ch}^{-2} x, \end{gathered} \tag{2.3}$$

to which it is necessary to join initial conditions

$$\tau = 0, \ \xi = \xi_0, \ \theta = 0, \ q = \alpha, \ \omega = \Omega(\alpha) \ (\lambda = 0), \tag{2.4}$$

and α plays the role of a parameter for the initial band ($\alpha \in R^1$). Initial data are obtained from conditions for conjugation of phase functions in expansions (1.3) and (2.1). Curves $\tau = \tau(\lambda, \alpha)$, $\xi = \xi(\lambda, \alpha)$, $\lambda > 0$ with a fixed value of α ($\alpha \in R^1$) are called rays.

Relationships (2.2)-(2.4) give

$$\omega(\lambda, \alpha) = \Omega(\alpha),\ q(\lambda, \alpha)\ \text{th}\ [q(\lambda, \alpha)h(\xi(\lambda, \alpha))] = \Omega^2(\alpha),$$

and $q(\lambda, \alpha) = 0$ only with $\alpha = 0$. If $|\alpha| > 0$ and $h > 0$, then $d\xi/d\lambda \neq 0, \pm\infty$, and it is possible to change from variable λ to variable ξ. Functions in which substitution $\lambda = \lambda(\xi, \alpha)$ is carried out are marked with index 1, for example $\tau = \tau_1(\xi, \alpha)$. In new variables problem (2.3), (2.4) is resolved in quadratures

$$\begin{gathered}\tau = \tau_1(\xi, \alpha) = -2\Omega(\alpha)\int_{\xi_0}^{\xi} S^{-1}[q_1(\beta, \alpha)h(\beta)]\,d\beta,\\ \theta = \theta_1(\xi, \alpha) = \Omega(\alpha)\tau + \int_{\xi_0}^{\xi} q_1(\beta, \alpha)\,d\beta,\\ q_1(\xi, \alpha)\ \text{th}\ [q_1(\xi, \alpha)h(\xi)] = \Omega^2(\alpha)\ (\xi \in R^1,\ \alpha \in R^1\setminus\{0\}).\end{gathered} \tag{2.5}$$

The following first approximation for expansion (2.1) leads to a transfer equation for the amplitude function

$$dC/d\lambda + k(\lambda, \alpha)C = 0\ (\lambda > 0),$$

where $C(\lambda, \alpha) = C_0(\xi(\lambda, \alpha), \tau(\lambda, \alpha), \xi_0)$,

$$k(\lambda, \alpha) = \theta_{\xi\xi}\left\{\frac{1}{4\Omega^2(\alpha)}S^2(qh(\xi)) + hS(qh)\,\text{th}\,qh - h\right\} - h_\xi q + \frac{q^2h_\xi}{\text{ch}^2\,qh}\left\{\frac{1}{4\Omega^2(\alpha)}S(qh) + h\,\text{th}\,qh\right\}.$$

Here $\theta_{\xi\xi} = q_\xi = dq_1(\xi, \tilde\alpha(\xi, \tau_{10}))/d\xi$, where $\tilde\alpha(\xi, \tau_{10})$ is such a function that $\tau_1(\xi, \tilde\alpha(\xi, \tau_{10})) = \tau_{10}$ = const. Consequently, $\theta_{\xi\xi} = q_{1\xi} + \tilde\alpha_\xi q_{1\alpha}$. By differentiating the identity determining $\tilde\alpha(\xi, \tau_{10})$ with respect to ξ, we obtain $\tilde\alpha_\xi = -\tau_{1\xi}/\tau_{1\alpha}$. In addition, from (2.5) we have

$$\begin{gathered}\partial q_1/\partial\xi = -q_1^2h_\xi[S(q_1h)\,\text{ch}^2(q_1h)]^{-1},\\ \partial q_1/\partial\alpha = 2\Omega\dot\Omega S^{-1}(q_1h),\ \ \partial\tau_1/\partial\xi = -2\Omega(\alpha)S^{-1}(q_1h),\\ \partial\tau_1/\partial\alpha = -2\dot\Omega\int_{\xi_0}^{\xi}\frac{d\beta}{S[q_1(\beta, \alpha)h(\beta)]} + 4\Omega^2\dot\Omega\int_{\xi_0}^{\xi}\frac{\dot S(q_1h)}{S^3(q_1h)}h(\beta)\,d\beta.\end{gathered} \tag{2.6}$$

The asymptotics constructed cease to be valid in the vicinity of those points (ξ, τ) for which the determinant of a Jacobi matrix $J = \partial(\xi, \tau)/\partial(\xi, \alpha)$ reduces to zero, i.e., $\partial\tau_1/\partial\alpha = 0$. In this case $\tilde\alpha_\xi$ reduces to infinity and the amplitude function $C_1(\xi, \alpha)$ becomes indefinitely large. The condition $|J| = 0$ means that in plane ξ, τ only two rays are found determined by the first equality of (2.5) and emerging from the initial ray at different angles which intersect with $\lambda > 0$. In the vicinity of this point representation $\xi, \alpha \to \xi, \tau$ ceases to be single-valued, which is also reflected in the equality $\partial\tau_1/\partial\alpha = 0$. In addition, since all of the rays emerge from one point $\xi = \xi_0$, $\tau = 0$, then the value of $\lambda = 0$ will also be singular.

With $\xi \to \xi_0$ Eqs. (2.6) give

$$\begin{gathered}\theta_{\xi\xi} = \tilde\alpha_\xi + O(1),\ \tau_{1\xi} = -2\Omega(\alpha)/S_0 + O(|\xi - \xi_0|),\\ \tau_{1\alpha} = (-1/(\Omega(\alpha)) + 2\Omega(\alpha)h_0\dot S_0/S_0^2)(\xi - \xi_0) + O(|\xi - \xi_0|^2),\\ S_0 = S(\alpha h_0),\end{gathered}$$

whence $k_1(\xi, \alpha) = -(1/2)S_0(\xi - \xi_0)^{-1} + O(1)$. Taking account of these asymptotics we write an equation for $C_1(\xi, \alpha)$:

$$C_{1\xi} + [(1/2)(\xi - \xi_0)^{-1} + D(\xi, \alpha)]C_1 = 0,$$

here $D(\xi, \alpha) = -k_1(\xi, \alpha)/S(q_1 h(\xi)) - (1/2)(\xi - \xi_0)^{-1}$ and $D = O(1)$ with $\xi \to \xi_0$. The general solution gives an equation

$$C_1(\xi, \alpha) = C_*(\alpha)\,|\xi - \xi_0|^{-1/2} \exp\left\{-\int_{\xi_0}^{\xi} D(\beta, \alpha)\, d\beta\right\}. \tag{2.7}$$

It is noted that $\tau = |\xi - \xi_0|/|\dot{\Omega}| + O(|\xi - \xi_0|^2)$ with $\xi \to \xi_0$, and therefore the requirement of agreement of amplitude functions in expansions (1.3) and (2.1) leads to an equation for $C_*(\alpha)$

$$C_*(\alpha) = \Omega^{-1}(\alpha)\,(|\dot{\Omega}(\alpha)|/(8\pi|\ddot{\Omega}(\alpha)|))^{1/2} e^{i\pi/4} \operatorname{ch}^{-1}\alpha h_0.$$

The main term of asymptotic expansion (2.1) has the form

$$U(x, y, t, x_0) = \varepsilon^{1/2} C_*(\alpha)\,|\xi - \xi_0|^{-1/2} \operatorname{ch} q_1(\xi, \alpha)(y + h(\xi)) \times$$
$$\times \exp\left\{\frac{i}{\varepsilon}\left(\Omega(\alpha)\tau + \int_{\xi_0}^{\xi} q_1(\beta, \alpha)\, d\beta\right) - \int_{\xi_0}^{\xi} D(\beta, \alpha)\, d\beta + i\frac{\pi}{4}\right\} + O(\varepsilon^{3/2}) + \text{c.c.} \tag{2.8}$$

Note 1. If $h(\xi) = h_0$ with $-a < \xi - \xi_0 < b$, $a > 0$, $b > 0$, then in this section $q_1 \equiv \alpha$, $D = 0$ and Eq. (2.8), as might be expected, conforms with (1.3). In addition, if in a certain interval $I \subset R^1$, $h_\xi(\xi) \equiv 0$, $\xi \in I$, then

$$U(\xi/\varepsilon, y, \tau/\varepsilon, x_0) = \sqrt{(c - \xi_0)/(\xi - \xi_0)}\, U(c/\varepsilon, y, \tau/\varepsilon, x_0) + O(\varepsilon^{3/2}),$$

(c is arbitrary point from $\bar{I}$).

Note 2. Proceeding from explicit Eq. (2.6), for the derivative $\partial\tau_1/\partial\alpha$ it is possible to show that asymptotic expansion (2.1) loses force not only in the vicinity of point $\xi = \xi_0$, $\tau = 0$, but also close to rays for which $\alpha = \pm 0$. With all of the rest of the values of ξ and τ representation $\xi, \tau \to \xi, \alpha$ is neutrally singular, since the integrand in the equation $\partial\tau_1/\partial\alpha$ is of fixed sign. With $\alpha \to 0$ $\partial\tau_1/\partial\alpha \to 0$, $\ddot{\Omega}(\alpha) \to 0$, and consequently it loses its applicability not only for expansion (2.1), but also for asymptotics (1.3). For an even bottom values $\alpha = \pm 0$ relate to waves propagating with critical velocity $\xi = \xi_0 \pm \sqrt{h_0}\tau$.

3. Right asymptotic behavior of (1.2) may be found by means of a generalized stationary phase method since $\dddot{\Omega}(0) \neq 0$. It is convenient to deal with not only potential U, but with its derivative U_t for which the asymptotic expansion is written out in terms of special functions. Following from the Whitham method [7] we obtain

$$U_t^{(0)}(x_1, 0, t, x_0) = -\frac{1}{2(3\gamma t)^{1/3}}\left\{\operatorname{Ai}\left(\frac{x_1 - t\sqrt{h_0}}{(3\gamma t)^{1/3}}\right) + \operatorname{Ai}\left(-\frac{x_1 + t\sqrt{h_0}}{(3\gamma t)^{1/3}}\right)\right\} + \ldots \tag{3.1}$$

with $t \to \infty$, $x_1/t \to 0$, $y = 0$, $\gamma = (1/6)h_0{}^{5/2}$. Here the first term corresponds to a wave propagating to the right. The Airy integral $\operatorname{Ai}(z)$ involved in (3.1) is

$$\operatorname{Ai}(z) = \frac{1}{\pi}\int_0^\infty \cos\left(zs + \frac{1}{3}s^3\right) ds.$$

With large values of the argument

$$\operatorname{Ai}(z) \sim \begin{cases} \dfrac{1}{2\sqrt{\pi}} z^{-1/4} \exp\left(-\dfrac{2}{3}z^{3/2}\right) & z \to +\infty, \\ \dfrac{1}{\sqrt{\pi}} |z|^{-1/4} \sin\left(\dfrac{2}{3}|z|^{3/2} + \dfrac{\pi}{4}\right) & z \to -\infty. \end{cases}$$

Whence it follows that the solution decreases exponentially ahead of the wave front $x_1 = \sqrt{h_0}t$ and it becomes oscillatory behind it. The transition region has a width proportional to $t^{1/3}$. Outside this region asymptotics (3.1) agree with the previously constructed (1.3).

In the transition region relating to a wave propagating to the right, instead of expansion (2.1) we use

$$-U_t(x, y, t, x_0) = \varepsilon^{1/3}\sum_{j=0}^{\infty}\varepsilon^j\left(A_j(\xi, y, \tau, \xi_0)\,\mathrm{Ai}(\zeta)+\varepsilon^{1/3}B_j(\xi, y, \tau, \xi_0)\,\mathrm{Ai}'(\zeta)\right), \tag{3.2}$$

where $\xi = \varepsilon x$, $\tau = \varepsilon t$, $\xi_0 = \varepsilon x_0$, $\zeta = \varepsilon^{-2/3}\rho(\xi, \tau)$. The form of this expansion follows from the same arrangements which were used in writing (2.1) and conditions for conformity with expansion (3.1) in the overlapping region. An expansion of the form (3.2) in which there is additionally a high frequency factor was used in [8] for constructing an approximation correct in the vicinity of a caustic.

Substitution of (3.2) in relationship (1.1) previously differentiated with respect to t and written in variables ξ, τ, y, and equating coefficients with $\varepsilon^j\mathrm{Ai}(\zeta)$, $\varepsilon^{j+1/3}\mathrm{Ai}'(\zeta)$ taking account of the equalities $\mathrm{Ai}''(\zeta) = \zeta\mathrm{Ai}(\zeta)$, $\varepsilon^{2/3}\zeta = \rho(\xi, \tau)$ leads to an equation for function $\rho(\xi, \tau)$ and a recurrent sequence of transfer equations for A_j, B_j, and the form of these equations depends on the sign of $\rho(\xi, \tau)$.

We consider a region in plane ξ, τ in which $\rho(\xi, \tau) > 0$. The curve $\rho(\xi, \tau) = 0$ corresponds to the crest of a leading wave and it should be determined in the course of solving the problem. In a zero approximation we have a spectral problem

$$A_{0yy} + \Phi_\xi^2 A_0 = 0\ (-h(\xi) < y < 0),\ A_{0y} = 0\ (y = -h(\xi)),$$
$$A_{0y} + \Phi^2_\tau A_0 = 0\ (y = 0),$$

whose nontrivial solution exists with the conditions

$$\Phi_\tau^2 = \Phi_\xi \operatorname{tg}\Phi_\xi h(\xi),\ \Phi(\xi, \tau) = (2/3)\rho^{3/2}(\xi, \tau) \tag{3.3}$$

and is written in the form

$$A_0(\xi, y, \tau, \xi_0) = C_0(\xi, \tau, \xi_0)\cos\Phi_\xi(y + h(\xi)).$$

By acting on the plan in part 1 we obtain a characteristic system relating to Eq. (3.3):

$$d\omega/d\lambda = 0,\ dq/d\lambda = q^2h_\xi/(\cos^2 qh),\ d\xi/d\lambda = -P(qh),\ d\tau/d\lambda = 2\omega,$$
$$d\Phi/d\lambda = 2\omega^2 - qP(qh),\ P(x) = \operatorname{tg} x + x/\cos^2 x \tag{3.4}$$

($\omega = \Phi_\tau$, $q = \Phi_\xi$). To this system we join the initial conditions

$$\xi = \xi_0,\ \tau = 0,\ \omega = \chi,\ q = q_0(\chi),\ \Phi = 0\ (\lambda = 0). \tag{3.5}$$

The first two equalities indicate that rays of Eq. (3.3) emerge from a single point, the third gives parametrization of the initial band, and the fourth and fifth follow from requirements of conformity for starting data. System (3.4) may be integrated if moving from λ, χ, to new independent variables ξ, χ. Functions τ, q, Φ in which such a change is carried out are marked as before with index 1. Then

$$\tau_1(\xi, \chi) = -2\chi\int_{\xi_0}^{\xi}\frac{d\xi}{P(q_1h)},\quad \Phi_1(\xi, \chi) = \chi\tau_1 + \int_{\xi_0}^{\xi} q_1(\xi, \chi)\,d\xi \tag{3.6}$$

[$q_1(\xi, \tau)$ is determined from the equation $q_1\tan q_1h(\xi) = \chi^2$, and $\operatorname{sgn} q_1 = -\operatorname{sgn}\chi$]. Thus, knowing the shape of the bottom of a body of water it is possible to determine the solution of problem (3.4), (3.5) in a parametric form by Eqs. (3.6). By expressing from the first of Eqs. (3.6) χ in terms of ξ, τ and substituting in the second equation we derive a dependence of ρ on variables ξ, τ. It is sufficient to know function $\rho(\xi, \tau)$ with small values of $\chi(\chi > 0)$, since with $\rho \sim 1$ we have $\zeta \sim \varepsilon^{-2/3}$ and $\mathrm{Ai}(\zeta) \sim e^{-1/\varepsilon}$ with $\varepsilon \to 0$. By expanding the right-hand parts of Eqs. (3.6) into a series with $\chi \to 0$ and discarding terms of the order $O(\chi^4)$ and above, we obtain

$$\rho(\xi, \tau) = 2^{1/3}\left(\int_{\xi_0}^{\xi} h^{-1/2}(\beta)\,d\beta - \tau\right)\left(\int_{\xi_0}^{\xi} h^{1/2}(\beta)\,d\beta\right)^{-1/3} + \ldots$$

This equation occurs for rays emerging from point $\xi = \xi_0$, $\tau = 0$ at angles close to the critical value. It is clear that $\rho(\xi, \tau) = 0$ along a curve

$$\tau = \int_{\xi_0}^{\xi} h^{-1/2}(\beta)\, d\beta \text{ (for a level bottom - along the lines } \xi - \xi_0 = \sqrt{h_0}\tau).$$

The transfer equation for function $C_{01}(\xi, \chi)$ takes the form

$$\frac{dC_{01}}{d\xi} - \frac{\sqrt{\rho}\,\Omega_0}{P(q_1 h)} C_{01} = 0, \tag{3.7}$$

where

$$\sqrt{\rho}\,\Omega_0 = \Phi_{\xi\xi}\left(\frac{1}{4\chi^2} P^2(q_1 h) - h - hP(q_1 h)\operatorname{tg} q_1 h\right) - \frac{1}{2\rho^{3/2}}\Phi_\tau^2 +$$
$$+ \frac{1}{4\rho^{3/2}}\Phi_\xi P(q_1 h) + \frac{1}{4\chi^2} P(q_1 h)\, q_1^2 h_\xi \cos^{-2} q_1 h - h \operatorname{tg} q_1 h \frac{q_1^2 h_\xi}{\cos^2 q_1 h} - h_\xi q_1.$$

The initial condition for Eq. (3.7) follows from the condition of conformity for expansions (1.3) and (3.2). It is noted that

$$\sqrt{\rho}\Omega_0/P(q_1 h) = -(1/3)(\xi - \xi_0)^{-1} + O(1) \quad (\xi \to \xi_0,\ \tau \to 0),$$

therefore the general solution of (3.7) may be written as

$$C_{01}(\xi, \chi) = C_0(\chi)(\xi - \xi_0)^{-1/3} \exp\left(-\int_{\xi_0}^{\xi} D(\beta, \chi)\, d\beta\right)$$

($D(\xi, \chi) = -\sqrt{\rho}\Omega_0/P(q_1 h) - (1/3)(\xi - \xi_0)^{-1}$). The joining condition for asymptotic expansions (1.3) and (3.2) gives $C_*(\chi) = (4h_0)^{-1/3} + o(1)$ with $\chi \to 0$.

Construction of asymptotics for the solution in region $\rho(\xi, \tau) < 0$ is accomplished similarly. As a result of this we obtain asymptotics for deformation of the free boundary $\eta(x, t)$ in the region of the leading wave

$$\eta(x, t) = (4h_0)^{-1/3}(h(\varepsilon x)/h_0)^{-1/4}\left(\int_{x_0}^{x} [h(\varepsilon\lambda)/h_0]^{1/2} d\lambda\right)^{-1/3} \times$$
$$\times \operatorname{Ai}\left\{2^{1/3}\left(\int_{x_0}^{x} h^{-1/2}(\varepsilon\lambda)\, d\lambda - t\right)\left(\int_{x_0}^{x} h^{1/2}(\varepsilon\lambda)\, d\lambda\right)^{-1/3}\right\} + O\left(\varepsilon^{1/3}\chi + \varepsilon^{2/3}\right) \tag{3.8}$$

with $\varepsilon \to 0$, $\chi\varepsilon^{-1/3} \sim 1$.

If $h = h_0$ in section $x_0 < x < x_1$, then Eqs. (3.8) and (3.1) give the same result. Let $x_0 = 0$ and $h(\varepsilon x) = A\varepsilon x$ with $x > x_1$, and $x_1 = h_0/A\varepsilon$, then the height of the tip of the leading wave calculated by Eq. (3.8) is

$$\eta(x_*(t), t) = 4^{1/12} 3^{1/3} \varepsilon^{1/3} h_0^{1/12} A^{-1/6} (\varepsilon t + h_0^{1/2}/A)^{-1/2} [h_0^{3/2} + (1/4) A^3 (\varepsilon t + h_0^{1/2}/A)^3]^{-1/3} \operatorname{Ai}(0) + \ldots,$$
$$x_*(t) = (A/4\varepsilon)(\varepsilon t + h_0^{1/2}/A)^2 \quad (\varepsilon t \geqslant \sqrt{h_0}/A,\ A > 0).$$

It can be seen that with emergence into the sloping section of the bottom the leading wave accelerates ($x_*(t) \sim t^2$, $t \to \infty$, for a level bottom $x_*(t) \sim t$), and the amplitude of its damping rate grows ($\eta \sim t^{-3/2}$ with $t \to \infty$, for an even bottom $\eta \sim t^{-1/3}$). These effects are strengthened with an increase in slope of the bottom.

4. Constructed above was the main term of asymptotics for solving problem (1.1) and a method was indicated for calculating the next approximations. The difficulty in constructing higher approximations is connected with the fact that in representation (3.2) functions $\rho(\xi, \tau)$, $A_0(\xi, y, \tau, \xi_0)$, $B_0(\xi, y, \tau, \xi_0)$ have finite smoothness in the vicinity of curve

$\tau = \int\limits_{\xi_0}^{\xi} h^{-1/2}(\beta)\,d\beta$, which corresponds to the region of long waves. In fact, these functions have a different form depending on from which direction on this curve we find ourselves. Therefore, in order to construct higher approximations it is necessary to refine in addition the solution in the region of long waves. The main term of asymptotics for the solution consists of three parts each of which is valid in its own region of the determination. However, these regions are overlapping zones and the overlap region $x \in R^1$, $-h(\varepsilon x) < y < 0$, $t > 0$ without gaps. Existence of overlapping zones makes it possible by known methods (the method of composite asymptotic expansions, the method of shearing functions, etc.) [9] to construct uniformly a convenient approximate solution which on being placed in relationship (1.1) gives a discrepancy; its order with respect to ε is unknown. In the case of localized unevenness of the bottom ($h(\xi) \equiv 1$ with $\xi \notin (\xi_1, \xi_2)$, $\varepsilon x_0 < \xi_1$) the order of the discrepancy is $\varepsilon^{3/2}$ with $\varepsilon \to 0$.

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